

Problem 3: Limit maps - Solution

First note that the recursive function converges to the point x . One can show this by solving the equation

$$y = 1 + \frac{x^2 - 1}{1 + y}.$$

This gives

$$y^2 = x^2.$$

So for positive $0 < x < 1$, this recursive formula converges to x .

Since we take x, y to be rational, we know the limit points of f_n (which are x, y , resp. as we have just shown) are also rational numbers between 0 and 1.

Note that the map Γ is periodic for rational numbers. It remains to find the correct periods of this map. One can also just repeatedly use this map, but we make sure that k is too big to do it in this way. If one knows the period T of the point (x, y) , one can take $k \bmod T$ to reduce the number of computations.

Since we are using rational numbers, it is important to write code that does not make round off errors. See the attached code.

Eventually, we look at the remainder r of $k \bmod T$, and we calculate $\Gamma^r(x, y)$ and return the correct integers that represent these rational numbers.